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Path integral representations for a system constrained on a manifold

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Abstract

A systematic study is made of path integral representations for a particle constrained to move on a manifold diffeomorphic to S^D by applying the irreducible representations of the Dirac algebra on it. Especially, we derive two types of path integral representation for this system, one of which is of a new form with a simple and compact expression, and the other is a rigorous version of the Faddeev–Senjanovic formula. It is also shown that the parameter $\alpha \in [0, 1)$ specifying the irreducible representation for $D = 1$ has a close connection with the Aharonov–Bohm gauge potential produced by the magnetic flux $\Phi = -2\pi\alpha\hbar c/e$.

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1. Introduction

We consider a quantum-mechanical system constrained to move on a D-dimensional manifold, which is embedded in the flat space $\mathbb{R}^{D+1}$. It will be denoted as $f(x) = 0$ with real $f(x)$ and is assumed to be diffeomorphic to S^D, where x stands for the coordinates x_α ($\alpha = 1, 2, \ldots, D + 1$) in $\mathbb{R}^{D+1}$. We also assume the Hamiltonian of the system to be given by

$$\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{x}) \qquad \text{with} \qquad \hat{p}^2 \equiv \hat{p}_\alpha \hat{p}_\alpha \tag{1.1}$$

where and in what follows repeated Greek indices indicate the summation over $1, 2, \ldots, D+1$. Then following Dirac [1] we may introduce the fundamental algebra for the operators $\hat{x}_\alpha$ and $\hat{p}_\alpha$ to guarantee a consistent description of the constrained system. The algebra will be called the Dirac algebra on $f(x) = 0$ and is shown to take the following form [2]:

$$f(\hat{x}) = 0 \tag{1.2}$$

$$\{\hat{p}_\alpha, f_{,\alpha}(\hat{x})\} = 0 \tag{1.3}$$

$$[\hat{x}_\alpha, \hat{x}_\beta] = 0 \tag{1.4}$$

$$[\hat{x}_\alpha, \hat{p}_\beta] = \mathrm{i}\hbar\Lambda_{\alpha\beta}(\hat{x}) \tag{1.5}$$

$$[\hat{p}_\alpha, \hat{p}_\beta] = -\mathrm{i}\hbar\left\{\frac{f_{,\alpha}(\hat{x})f_{,\beta\gamma}(\hat{x}) - f_{,\beta}(\hat{x})f_{,\alpha\gamma}(\hat{x})}{2R^2(\hat{x})}, \hat{p}_\gamma\right\} \qquad (\alpha, \beta, \gamma = 1, 2, \ldots, D+1) \tag{1.6}$$

where

$$f_{,\alpha}(x) = \partial_\alpha f(x) \qquad f_{,\alpha\beta}(x) = \partial_\alpha\partial_\beta f(x) \tag{1.7}$$

and $R(x)$ and $\Lambda_{\alpha\beta}(x)$ are defined, respectively, by

$$R(x) = (f_{,\alpha}(x)f_{,\alpha}(x))^{1/2} \qquad \Lambda_{\alpha\beta}(x) = \delta_{\alpha\beta} - \frac{f_{,\alpha}(x)f_{,\beta}(x)}{R^2(x)}. \tag{1.8}$$

It is noted here that $R(x)$ is non-vanishing in the neighbourhood of $f(x) = 0$ because of diffeomorphism between $f(x) = 0$ and S^D.

Recently all possible irreducible representations of the above algebra (1.2)–(1.6) have been determined completely [2]. They are represented in terms of the canonical variables $\hat{x}_\alpha$ and $\hat{\pi}_\alpha$ ($\alpha = 1, 2, \ldots, D+1$) that satisfy

$$[\hat{x}_\alpha, \hat{x}_\beta] = [\hat{\pi}_\alpha, \hat{\pi}_\beta] = 0 \qquad [\hat{x}_\alpha, \hat{\pi}_\beta] = \mathrm{i}\hbar\delta_{\alpha\beta}. \tag{1.9}$$

With the aid of these we will in the present paper explicitly construct path integral representations for a system constrained on $f(x) = 0$. Thus, for the sake of later convenience, we will first of all summarize the main results obtained in our previous work [2] (referred to as I). Throughout the present paper the notation and definitions are the same as those in I.

The operators $\hat{p}_\beta$ in the irreducible representation of the Dirac algebra are expressed as follows:

$D = 1$.

$$\hat{p}_\beta = \frac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}), \hat{\pi}_\gamma\} - \alpha\hbar\frac{\Lambda_{\beta\gamma}(\hat{x})f_{,\gamma\rho}(\hat{x})f_{,\sigma}(\hat{x})\epsilon_{\rho\sigma}}{R^2(\hat{x})} \tag{1.10}$$

where α is a real parameter which uniquely specifies the irreducible representation, and $\epsilon_{\rho\sigma}$ stands for the two-dimensional Levi-Civita symbol defined by $\epsilon_{\rho\sigma} = -\epsilon_{\sigma\rho}$ and $\epsilon_{12} = 1$. It has also been shown that for $D = 1$ (i) no other irreducible representation exists than the above and (ii) two irreducible representations specified by α and α', respectively, are unitarily equivalent if and only if $\alpha' = \alpha$ + integer. Hence without loss of generality we can restrict ourselves to the cases for $0 \leqslant \alpha < 1$.

$D \geqslant 2$.

$$\hat{p}_\beta = \tfrac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}), \hat{\pi}_\gamma\} \tag{1.11}$$

which is unique except for unitary-equivalent representations.

Given an irreducible representation we denote the representation space as $\underline{\mathcal{H}}$ and state vectors belonging to it as $|\underline{\psi})$, $|\underline{\chi})$, ... Corresponding to them we write the wavefunctions as $\underline{\psi}(\underline{x}) \equiv (\underline{x}|\underline{\psi})$, $\underline{\chi}(\underline{x}) \equiv (\underline{x}|\underline{\chi})$, ..., where and in what follows $\underline{x}$ stands for a point on $f(x) = 0$. We further introduce auxiliary wavefunctions $\psi(x)$, $\chi(x)$, ... belonging to L^2 on $\mathbb{R}^{D+1}$. They are required to satisfy

$$\begin{cases} \psi(x)|_{x=\underline{x}} = \underline{\psi}(\underline{x}) \\ \chi(x)|_{x=\underline{x}} = \underline{\chi}(\underline{x}) \\ \quad\vdots \qquad\qquad \vdots \end{cases}. \tag{1.12}$$

Then the inner product and the matrix element of an operator $O(\hat{x}, \hat{p})$ on $\underline{\mathcal{H}}$ can be written as

$$(\underline{\psi}|\underline{\chi}) = \int \mathrm{d}^{D+1}x \, \delta(f(x))\psi^*(x)\chi(x) \tag{1.13}$$

and

$$\begin{aligned}(\underline{\psi}|O(\hat{x}, \hat{p})|\underline{\chi}) &= \int \mathrm{d}^{D+1}x \, \mathrm{d}^{D+1}x' \delta(f(x))\psi^*(x)\langle x|O(\hat{x}, \hat{p})|x'\rangle\chi(x') \\ &= \int \mathrm{d}^{D+1}x \, \mathrm{d}^{D+1}x' \psi^*(x)\langle x|O(\hat{x}, \hat{p})|x'\rangle\delta(f(x'))\chi(x')\end{aligned} \tag{1.14}$$

with the normalized $f(x)$ such that

$$R(\underline{x}) = (f_{,\alpha}(\underline{x})f_{,\alpha}(\underline{x}))^{1/2} = 1. \tag{1.15}$$

As emphasized in I, condition (1.15) is crucial in defining the inner product in the form of (1.13). In (1.14) the ket $|x\rangle$ stands for the eigenstate of the position operators $\hat{x}_\alpha$ in $\mathbb{R}^{D+1}$ that satisfy (1.9), and hence

$$\hat{x}_\alpha|x\rangle = x_\alpha|x\rangle \qquad \langle x|x'\rangle = \delta^{D+1}(x - x') \qquad \int \mathrm{d}^{D+1}x|x\rangle\langle x| = \hat{1} \tag{1.16}$$

where $\hat{1}$ is the unit operator on the representation space $\mathcal{H}$ of canonical commutation relations (1.9), and each of the spectra x_α ranges from $-\infty$ to ∞. It is noted that according to (1.10) and (1.11) any operator $O(\hat{x}_\alpha, \hat{p}_\alpha)$ on $\underline{\mathcal{H}}$ is also an operator on $\mathcal{H}$. Thus arguments about the constrained system under consideration are reduced to those in the usual case described by (1.9).

Consequently we can express the transition amplitude T_{FI} from the initial state $|\psi_I)$ at $t = t_I$ to the final state $|\psi_F)$ at $t = t_F$ as

$$\begin{aligned}T_{FI} &= (\underline{\psi}_F|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t_F - t_I)\right)|\underline{\psi}_I) \\ &= \int \mathrm{d}^{D+1}x_F \, \mathrm{d}^{D+1}x_I \delta(f(x_F))\psi_F^*(x_F)\langle x_F|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t_F - t_I)\right)|x_I\rangle\psi_I(x_I)\end{aligned} \tag{1.17}$$

which enables us to obtain the amplitude T_{FI} by calculating the propagation function $\langle x_F|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t_F - t_I)\right)|x_I\rangle$ on $\mathcal{H}$. Needless to say $\psi_I(x_I)$ and $\psi_F(x_F)$ are auxiliary wavefunctions corresponding to $|\underline{\psi}_I)$ and $|\underline{\psi}_F)$, respectively. In other words, since given an irreducible representation of the Dirac algebra the Hamiltonian $\hat{H}$ is expressed as a function of canonical variables $\hat{x}_\alpha$ and $\hat{\pi}_\alpha$, applying the usual technique in path integration [3] we can derive the path integral form of $\langle x_F|\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}(t_F - t_I)\right)|x_I\rangle$ and hence of T_{FI}.

It would be worthwhile to insert here a short remark. If we employ the function

$$f_c(x) \equiv f(x) - c \qquad (c : \text{real}) \tag{1.18}$$

in lieu of $f(x)$ in the Dirac algebra, then for small $|c|$ the primary constraint $f_c(x) = 0$ in this case provides us with a D-dimensional manifold in the neighbourhood of $f(x) = 0$. Since there holds $f_{c,\beta}(x) = f_{,\beta}(x)$ in this domain, the secondary constraint $\{\hat{p}_\alpha, f_{c,\alpha}(\hat{x})\} = 0$ takes the same form as (1.3) and commutators (1.4)–(1.6) remain unchanged under $f(x) \to f_c(x)$. Furthermore, the operators $\hat{p}_\beta$ of (1.10) and (1.11) are seen to describe the irreducible representations of the Dirac algebra on $f_c(x) = 0$ together with $\hat{x}_\alpha$ that satisfy $f_c(\hat{x}) = 0$. Thus most of results obtained on $f(x) = 0$ are just generalized to those on $f_c(x) = 0$.

In the next section on the basis of (1.17) we try to derive a rigorous form of the path integral for the constrained system under consideration. To this end for the sake of definiteness we divide the time interval $t_F - t_I$ into N segments, each of which is $\Delta t = (t_F - t_I)/N$, and take the limit $N \to \infty$ after all calculations. Then we obtain a basic form of path integral representation for the system which includes quantum effects completely. Using this we derive a Lagrangian path integral representation called type I. By means of it, in section 2, we also discuss the physical meaning of the parameter α, which appears for $D = 1$, in a relation with the Aharonov–Bohm gauge potential [4].

In section 3, based on the argument in section 2, we try to rewrite the Faddeev [5] and Senjanovic [6] formulae applied to a system constrained on $f(x) = 0$ to obtain its rigorous and well-defined form. Related to this some quantum effects are examined. The Lagrangian path integral representation in this case, called type II, is also derived. It has a different appearance from type I.

The final section will be devoted to some additional remarks.

2. Path integral representations

To begin with let us rewrite Hamiltonian (1.1) in terms of the canonical variables $\hat{x}_\beta$ and $\hat{\pi}_\beta$ by applying (1.10) and (1.11). Then after some calculations we find

$$\hat{H} = \frac{1}{2}\hat{\pi}_\beta \Lambda_{\beta\gamma}(\hat{x})\hat{\pi}_\gamma + K(\hat{x}) + V(\hat{x}) + \frac{\alpha\hbar}{2}\epsilon_{\beta\tau}\left\{\frac{f_{,\beta}(\hat{x})f_{,\tau\sigma}(\hat{x})\Lambda_{\sigma\rho}(\hat{x})}{R^2(\hat{x})}, \hat{\pi}_\rho\right\} + \frac{\alpha^2\hbar^2}{2}\frac{\Lambda_{\beta\gamma}(\hat{x})f_{,\gamma\sigma}(\hat{x})\Lambda_{\sigma\tau}(\hat{x})f_{,\tau\beta}(\hat{x})}{R^2(\hat{x})} \qquad (D = 1) \tag{2.1}$$

and

$$\hat{H} = \tfrac{1}{2}\hat{\pi}_\beta \Lambda_{\beta\gamma}(\hat{x})\hat{\pi}_\gamma + K(\hat{x}) + V(\hat{x}) \qquad (D \geqslant 2) \tag{2.2}$$

where

$$K(\hat{x}) = -\frac{\hbar^2}{8}\left\{\partial_\beta\partial_\sigma\Lambda_{\beta\sigma}(\hat{x}) - \frac{\Lambda_{\alpha\beta}(\hat{x})f_{,\alpha\rho}(\hat{x})f_{,\beta\rho}(\hat{x})}{R^2(\hat{x})}\right\}. \tag{2.3}$$

On the basis of (2.1) and (2.2) we will formulate path integral representations for the system under consideration. To this end we evaluate the term $\langle x^{(k)}|\hat{H}|x^{(k-1)}\rangle$ $(k = 1, 2, \ldots, N)$. As shown in appendix A we then obtain the following:

$$\langle x^{(k)}|\hat{H}|x^{(k-1)}\rangle = \int \frac{\mathrm{d}^{D+1}p^{(k)}}{(2\pi\hbar)^{D+1}} \exp\left[\frac{\mathrm{i}}{\hbar}p^{(k)}\Delta x^{(k)}\right] H(p^{(k)\perp}, \bar{x}^{(k)}) \tag{2.4}$$

with

$$H(p^{(k)\perp}, \bar{x}^{(k)}) = \frac{1}{2}(p^{(k)\perp})^2 + V_{\mathrm{eff}}(\bar{x}^{(k)}) + \alpha\hbar\frac{\epsilon_{\beta\tau}f_{,\beta}(\bar{x}^{(k)})f_{,\tau\sigma}(\bar{x}^{(k)})p_\sigma^{(k)\perp}}{R^2(\bar{x}^{(k)})} + \alpha^2\hbar^2\frac{\Lambda_{\beta\gamma}(\bar{x}^{(k)})f_{,\gamma\sigma}(\bar{x}^{(k)})\Lambda_{\sigma\tau}(\bar{x}^{(k)})f_{,\tau\beta}(\bar{x}^{(k)})}{2R^2(\bar{x}^{(k)})} \qquad (D = 1) \tag{2.5}$$

and

$$H(p^{(k)\perp}, \bar{x}^{(k)}) = \tfrac{1}{2}(p^{(k)\perp})^2 + V_{\mathrm{eff}}(\bar{x}^{(k)}) \qquad (D \geqslant 2) \tag{2.6}$$

where

$$p_\beta^{(k)\perp} = \Lambda_{\beta\gamma}(\bar{x}^{(k)})p_\gamma^{(k)} \qquad \bar{x}_\beta^{(k)} = \frac{x_\beta^{(k)} + x_\beta^{(k-1)}}{2} \qquad \Delta x_\beta^{(k)} = x_\beta^{(k)} - x_\beta^{(k-1)}$$

and

$$V_{\mathrm{eff}}(\bar{x}^{(k)}) = \hbar^2\frac{\Lambda_{\alpha\beta}(\bar{x}^{(k)})f_{,\alpha\rho}(\bar{x}^{(k)})f_{,\beta\rho}(\bar{x}^{(k)})}{8R^2(\bar{x}^{(k)})} + V(\bar{x}^{(k)}). \tag{2.7}$$

Thus with the aid of the well-known technique in path integral, we are led to

$$\langle x_F|\exp\left(-\frac{\mathrm{i}}{\hbar}(t_F-t_I)\hat{H}\right)|x_I\rangle = \lim_{N\to\infty}\int\prod_{k=1}^{N-1}\mathrm{d}^{D+1}x^{(k)}$$
$$\times\int\prod_{k=1}^{N}\frac{\mathrm{d}^{D+1}p^{(k)}}{(2\pi\hbar)^{(D+1)}}\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{k=1}^{N}\{p^{(k)}\cdot\Delta x^{(k)} - H(p^{(k)\perp},\bar{x}^{(k)})\Delta t\}\right] \tag{2.8}$$

where

$$t^{(N)}\equiv t_F \qquad t^{(0)}\equiv t_I \qquad \Delta t\equiv\frac{t_F-t_I}{N}. \tag{2.9}$$

Then applying (2.8) to the right-hand side of (1.17), we arrive at

$$T_{FI} = \lim_{N\to\infty}\int\prod_{k=0}^{N}\mathrm{d}^{D+1}x^{(k)}\cdot\delta(f(x^{(N)}))\int\prod_{k=1}^{N}\frac{\mathrm{d}^{D+1}p^{(k)}}{(2\pi\hbar)^{(D+1)}}$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{k=1}^{N}\{p^{(k)}\cdot\Delta x^{(k)} - H(p^{(k)\perp},\bar{x}^{(k)})\Delta t\}\right]\psi_F^*(x^{(N)})\psi_I(x^{(0)}) \tag{2.10}$$

which provides us with a rigorous and basic expression of the path integral for a system constrained on $f(x)=0$. The emergence of the argument $p^{(k)\perp}$ in the Hamiltonian of (2.10) seems quite reasonable, since in the continuous limit it describes a projection of the momentum p onto the space tangential to the manifold at $\bar{x}^{(k)}$. Though the expression of (2.10) is simple and compact, it apparently differs from the Faddeev–Senjanovic (FS) [5, 6] formula derived by a semi-classical approach. The relation of (2.10) with the FS formula will be discussed in the next section.

We are now ready to perform the p-integration in (2.10) using (2.5) for $D=1$ and (2.6) for $D\geqslant 2$. For later convenience we start by evaluating the integral such that

$$I(x,x') = \int\mathrm{d}^{D+1}p\exp\left[\frac{\mathrm{i}}{\hbar}\{p\cdot\Delta x - H(p^{\perp},\bar{x})\Delta t\}\right] \tag{2.11}$$

where

$$\Delta x\alpha = x\alpha - x\alpha' \qquad \bar{x}\alpha = \tfrac{1}{2}(x\alpha + x\alpha') \qquad p\alpha^{\perp} = \Lambda_{\alpha\beta}(\bar{x})p_\beta.$$

Since the contribution of $(\bar{x})$ to the path integral is considered to come from the neighbourhood of the manifold, we may regard the vector $f_{,\cdot}(\bar{x})\equiv(f_{,1}(\bar{x}), f_{,2}(\bar{x}),\ldots,f_{,D+1}(\bar{x}))$ as non-vanishing. Then we introduce an orthogonal transformation represented by the matrix $\|a_{\beta\gamma}(\bar{x})\|\in SO(D+1)$, which rotates the vector $f,.(\bar{x})$ to the direction of the $(D+1)$-th axis, i.e.,

$$a_{\beta\gamma}(\bar{x})f_{,\gamma}(\bar{x}) = \delta_{\beta D+1}R(\bar{x}). \tag{2.12}$$

Applying it we define the quantities such that

$$X_\beta = a_{\beta\gamma}(\bar{x})x_\gamma \qquad X'_\beta = a_{\beta\gamma}(\bar{x})x'_\gamma \qquad P_\beta = a_{\beta\gamma}(\bar{x})p_\gamma \qquad P_\beta^{\perp} = a_{\beta\gamma}(\bar{x})p_\gamma^{\perp}. \tag{2.13}$$

We will use a boldfaced letter for a D-dimensional vector obtained by dropping out the $(D+1)$-th component from a vector in $\mathbb{R}^{D+1}$, and hence, for example, we will denote it as $\boldsymbol{X}\equiv(X_1,X_2,\ldots,X_D)$ for $X=(X_1,X_2,\ldots,X_{D+1})$. Then there holds $P^{\perp}=(\boldsymbol{P},0)$. Denoting the Hamiltonian $H(p^{\perp},\bar{x})$ as $\boldsymbol{H}(\boldsymbol{P},\bar{x})$, we rewrite $I(x,x')$ as

$$I(x,x') = \int\mathrm{d}^{D}\boldsymbol{P}\mathrm{d}P_{D+1}\exp\left[\frac{\mathrm{i}}{\hbar}\{P\cdot\Delta X - \boldsymbol{H}(\boldsymbol{P},\bar{x})\Delta t\}\right]$$
$$= 2\pi\hbar\delta(\Delta X_{D+1})\tilde{I}(x,x') \tag{2.14}$$

with

$$\tilde{I}(x, x') = \int \mathrm{d}^D \boldsymbol{P} \exp\left[\frac{\mathrm{i}}{\hbar}\{\boldsymbol{P}\cdot\Delta\boldsymbol{X} - \boldsymbol{H}(\boldsymbol{P}, \bar{x})\Delta t\}\right] \tag{2.15}$$

where

$$\Delta X_\beta = X_\beta - X'_\beta = a_{\beta\gamma}(\bar{x})\Delta x_\gamma. \tag{2.16}$$

By definition the Hamiltonian $\boldsymbol{H}(\boldsymbol{P}, \bar{x})$ takes the following form:

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{P}, \bar{x}) = \frac{1}{2}\boldsymbol{P}^2 + V_{\mathrm{eff}}(\bar{x}) + \alpha\hbar\frac{\epsilon_{\sigma\tau} f_{,\sigma}(\bar{x}) f_{,\tau\beta}(\bar{x}) a_{1\beta}(\bar{x})\boldsymbol{P}}{R^2(\bar{x})} \\ + \alpha^2\hbar^2\frac{\Lambda_{\beta\gamma}(\bar{x}) f_{,\gamma\sigma}(\bar{x})\Lambda_{\sigma\tau}(\bar{x}) f_{,\tau\beta}(\bar{x})}{2R^2(\bar{x})} \qquad (D = 1)\end{aligned} \tag{2.17}$$

and

$$\boldsymbol{H}(\boldsymbol{P}, \bar{x}) = \tfrac{1}{2}\boldsymbol{P}^2 + V_{\mathrm{eff}}(\bar{x}) \qquad (D \geqslant 2). \tag{2.18}$$

It is noted that $\boldsymbol{P}$ in (2.17) is a single component vector in one dimension.

Thus by applying the Fresnel formula

$$\int \mathrm{d}^D\boldsymbol{P} \exp\left[-\frac{\mathrm{i}}{2\hbar}\boldsymbol{P}^2\Delta t\right] = \left(\frac{2\pi\hbar}{\mathrm{i}\Delta t}\right)^{D/2}$$

we can perform the $\boldsymbol{P}$-integration in (2.15) for Hamiltonians (2.17) and (2.18). Then we find that $\tilde{I}(x, x')$ can effectively be written as

$$\tilde{I}(x, x') = \left(\frac{2\pi\hbar}{\mathrm{i}\Delta t}\right)^{1/2} \exp\left[\frac{\mathrm{i}}{\hbar}\left\{\frac{1}{2}\left(\frac{\Delta x_\beta}{\Delta t}\right)^2 + A_\beta(\bar{x})\frac{\Delta x_\beta}{\Delta t} - V_{\mathrm{eff}}(\bar{x})\right\}\Delta t\right] \qquad (D = 1) \tag{2.19}$$

with

$$A_\beta(x) \equiv -\alpha\hbar\frac{\epsilon_{\sigma\tau} f_{,\sigma}(x) f_{,\tau\beta}(x)}{R^2(x)} \tag{2.20}$$

and

$$\tilde{I}(x, x') = \left(\frac{2\pi\hbar}{\mathrm{i}\Delta t}\right)^{D/2} \exp\left[\frac{\mathrm{i}}{\hbar}\left\{\frac{1}{2}\left(\frac{\Delta x_\beta}{\Delta t}\right)^2 - V_{\mathrm{eff}}(\bar{x})\right\}\Delta t\right] \qquad (D \geqslant 2). \tag{2.21}$$

The derivation of (2.19) is somewhat technical. It is given in appendix B.

Since transition amplitude (2.10) is written as

$$\begin{aligned}T_{FI} &= \lim_{N\to\infty}\int\prod_{k=0}^{N}\mathrm{d}^{D+1}x^{(k)}\cdot\delta(f(x^{(N)}))\prod_{k=1}^{N}\left(\frac{I(x^{(k)}, x^{(k-1)})}{(2\pi\hbar)^{D+1}}\right)\psi_F^*(x^{(N)})\psi_I(x^{(0)}) \\ &= \lim_{N\to\infty}\int\prod_{k=0}^{N}\mathrm{d}^{D+1}x^{(k)}\cdot\delta(f(x^{(N)}))\prod_{k=1}^{N}\left(\frac{\delta\big(\Delta X_{D+1}^{(k)}\big)\tilde{I}(x^{(k)}, x^{(k-1)})}{(2\pi\hbar)^D}\right)\psi_F^*(x^{(N)})\psi_I(x^{(0)})\end{aligned} \tag{2.22}$$

with the help of the relation

$$\Delta X_{D+1}^{(k)} = \Delta x_\beta^{(k)} f_{,\beta}(\bar{x}^{(k)})/R(\bar{x}^{(k)}) \tag{2.23}$$

we obtain

$$T_{FI} = \lim_{N\to\infty} \frac{1}{(2\pi i\hbar\Delta t)^{DN/2}} \prod_{k=0}^{N} \int d^{D+1}x^{(k)} \cdot \delta(f(x^{(N)})) \prod_{k=1}^{N} \delta\big(\Delta x_\beta^{(k)} f_{,\beta}(\bar{x}^{(k)}) \big/ R(\bar{x}^{(k)})\big)$$
$$\times \exp\left[\frac{i\Delta t}{\hbar}\sum_{k=1}^{N} L_{\text{eff}}^{(k)}\right] \psi_F^*(x^{(N)})\psi_I(x^{(0)}) \tag{2.24}$$

where the effective Lagrangian $L_{\text{eff}}^{(k)}$ is given by

$$L_{\text{eff}}^{(k)} = \frac{1}{2}\left(\frac{\Delta x_\beta^{(k)}}{\Delta t}\right)^2 + A_\beta(\bar{x}^{(k)})\frac{\Delta x_\beta^{(k)}}{\Delta t} - V_{\text{eff}}(\bar{x}^{(k)}) \qquad (D = 1) \tag{2.25}$$

and

$$L_{\text{eff}}^{(k)} = \frac{1}{2}\left(\frac{\Delta x_\beta^{(k)}}{\Delta t}\right)^2 - V_{\text{eff}}(\bar{x}^{(k)}) \qquad (D \geqslant 2). \tag{2.26}$$

Needless to say, in (2.24) we can substitute $\delta(f(x^{(0)}))$ for $\delta(f(x^{(N)}))$ by using (1.14). It is to be remarked that in (2.24) the factor $\delta\big(\Delta x_\beta^{(k)} f_{,\beta}(\bar{x}^{(k)})\big/R(\bar{x}^{(k)})\big)$, which corresponds to secondary constraint (1.3), emerges for each k, while the factor corresponding to primary constraint (1.2) appears once in the form of $\delta(f(x^{(N)}))$ or $\delta(f(x^{(0)}))$. We call (2.24) the path integral representation of type I.

In the continuous limit Lagrangian (2.26) for $D \geqslant 2$ becomes of the standard form with the effective potential V_{eff}, in which a quantum effect proportional to $\hbar^2$ has been brought through the process of ordering the canonical variables $\hat{x}_\alpha$ and $\hat{\pi}_\alpha$ in the kinetic energy part of the Hamiltonian.

In contrast to the case of $D \geqslant 2$, the Lagrangian for $D = 1$ has a kind of gauge interaction, which is reduced to the form $A_\beta(x)\dot{x}_\beta$ in the continuous limit. The situation is very characteristic of $D = 1$. Hence in the following we will examine properties of (2.20), which hereafter will be denoted in units of $e/c = 1$. Since the components of the gauge potential are of the form

$$\begin{cases} A_1(x) = \alpha\hbar \dfrac{f_{,2}(x)f_{,11}(x) - f_{,1}(x)f_{,12}(x)}{R^2(x)} \\ A_2(x) = \alpha\hbar \dfrac{f_{,2}(x)f_{,12}(x) - f_{,1}(x)f_{,22}(x)}{R^2(x)} \end{cases} \tag{2.27}$$

they are seen to satisfy

$$\frac{\partial A_1(x)}{\partial x_2} - \frac{\partial A_2(x)}{\partial x_1} = 0 \qquad (R(x) \neq 0), \tag{2.28}$$

that is, no magnetic flux exists in the neighbourhood of the closed loop $f(x) = 0$ on $\mathbb{R}^2$. Now let us introduce the quantity

$$\zeta(x) \equiv \tan^{-1}\left(\frac{f_{,2}(x)}{f_{,1}(x)}\right). \tag{2.29}$$

For x on $f(x) = 0$ it stands for the angle between the x_1-axis and the normal to the closed loop at x (see figure 1).

As easily seen gauge potential (2.27) is written as

$$A_\beta(x) = -\alpha\hbar\frac{\partial\zeta(x)}{\partial x_\beta} \qquad (R(x) \neq 0) \tag{2.30}$$

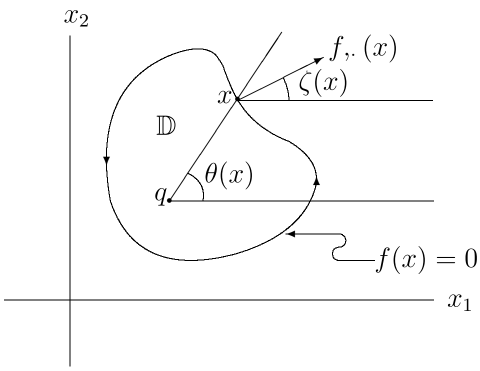

Figure 1. The domain $\mathbb{D}$ is surrounded by the closed loop $f(x) = 0$. The vector $f_{,\cdot}(x) = (f_{,1}(x), f_{,2}(x))$ is a normal to the closed loop at x.

and hence the contour integral $\oint A_\beta(x)\,\mathrm{d}x_\beta$ evaluated along the closed loop takes the value $-2\pi\alpha\hbar$, i.e. the gauge potential is non-trivial for $\alpha \neq 0$.

Now let us take a point $q = (q_1, q_2)$ inside the domain $\mathbb{D}$ which is surrounded by the closed loop $f(x) = 0$ on $\mathbb{R}^2$. Using it we further introduce the potential such that

$$\mathfrak{A}_\beta(x) = -\frac{1}{2\pi}\epsilon_{\beta\tau}\frac{x_\tau - q_\tau}{(x_\sigma - q_\sigma)^2} = \frac{1}{2\pi}\frac{\partial\theta(x)}{\partial x_\beta} \tag{2.31}$$

with

$$\theta(x) = \tan^{-1}\left(\frac{x_2 - q_2}{x_1 - q_1}\right) \qquad (x \neq q). \tag{2.32}$$

The potential $\mathfrak{A}_\beta(x)$ is nothing but the Aharonov–Bohm gauge potential [4] produced by the unit magnetic flux confined in an extremely thin solenoid perpendicular to $\mathbb{R}^2$ at q. As the point x circles once along the closed loop, the angles $\zeta(x)$ and $\theta(x)$ are transformed as

$$\zeta(x) \to \zeta(x) + 2\pi \qquad \theta(x) \to \theta(x) + 2\pi. \tag{2.33}$$

Consequently the function

$$U(x) = \exp\left[\frac{\mathrm{i}}{\hbar}F(x)\right] \tag{2.34}$$

defined with

$$F(x) = -\alpha\hbar(\zeta(x) - \theta(x)) \tag{2.35}$$

is a single-valued function in the neighbourhood of the closed loop. Then it enables us to make a gauge transformation such that

$$A_\beta(x) \to A_\beta(x) - \frac{\hbar}{\mathrm{i}}U^*(x)\frac{\partial U(x)}{\partial x_\beta} = -2\pi\alpha\hbar\mathfrak{A}_\beta(x). \tag{2.36}$$

As mentioned in section 1 the arguments on the closed loop $f(x) = 0$ can be generalized to those on $f_c(x) = 0$. Thus it is concluded that the potential $A_\beta(x)$ is definable in the neighbourhood of $f(x) = 0$ and is gauge equivalent to the Aharonov–Bohm potential produced by the magnetic flux $\Phi = -2\pi\alpha\hbar$ that perpendicularly crosses the domain $\mathbb{D}$ at q.

It is also shown that the parameter α is essentially the same as that introduced by Schulman [7] in his path integral formalism, which was derived without recourse to the operator formalism. The emergence of α originates in the multiply connected structure [8] of the closed loop.

3. Relation with the Faddeev–Senjanovic formula

The FS path integral formula [5, 6] for a system constrained on the D-dimensional manifold $f(x) = 0$ is written as

$$T_{FI} = \int D\mu \psi_F^*(x_F) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{T_I}^{T_F} \mathrm{d}t(p\dot{x} - H(p,x))\right]\psi_I(x_I) \tag{3.1}$$

with $H = \frac{1}{2}p^2 + V(x)$, where the integral measure $D\mu$ is given by

$$D\mu = \delta(f(x))\delta(p_\alpha f_{,\alpha}(x))R^2(x)DpDx. \tag{3.2}$$

Amplitude (3.1) was derived by applying a semi-classical argument based on the Dirac formalism [1] for the constrained system, since at that time nothing was known on the irreducible representation of the Dirac algebra. Accordingly there may arise questions, say, about quantum effects which could emerge in the path integral representation and also about a concrete definition of measure (3.2). In this connection Kashiwa and Fukutaka [9, 10] carefully examined formula (3.1) to determine a correct form of the path integral without use of irreducible representations.

In the following, starting from (2.10) we will derive a rigorous expression for the path integral that corresponds to (3.1). For this purpose, without performing the p-integration we rewrite $\tilde{I}(x, x')$ of (2.15) in the following manner:

$$\begin{aligned}\tilde{I}(x,x') &= \int \mathrm{d}^D \boldsymbol{P} \exp\left[\frac{\mathrm{i}}{\hbar}\{\boldsymbol{P}\cdot\Delta\boldsymbol{X} - \boldsymbol{H}(\boldsymbol{P},\bar{x})\Delta t\}\right] \\ &= \int \mathrm{d}^{D+1}P\delta(P_{D+1}R(\bar{x}))R(\bar{x})\exp\left[\frac{\mathrm{i}}{\hbar}\{P\cdot\Delta X - \boldsymbol{H}(P,\bar{x})\Delta t\}\right] \\ &= R(\bar{x})\int \mathrm{d}^{D+1}p\delta(p_\beta f_{,\beta}(\bar{x}))\exp\left[\frac{\mathrm{i}}{\hbar}\{p\cdot\Delta x - H(p,\bar{x})\Delta t\}\right] \end{aligned}\tag{3.3}$$

where $\boldsymbol{H}(\boldsymbol{P},\bar{x}) = H(p^\perp,\bar{x})$ and $H(p,\bar{x})$ is given by simply replacing $p_\beta^\perp$ with p_β in $H(p^\perp,\bar{x})$. The factor $\delta(p_\beta f_{,\beta}(\bar{x}))$ in the integrand just corresponds to the secondary constraint $\{\hat{p}_\beta, f_{,\beta}(\hat{x})\} = 0$.

To define the path integral measure corresponding to (3.2) we further rewrite the factor $\delta(\Delta X_{D+1})$ in (2.14). To this end we proceed in the following way:

$$\begin{aligned} f(x) - f(x') &= f\left(\bar{x} + \frac{\Delta x}{2}\right) - f\left(\bar{x} - \frac{\Delta x}{2}\right) \\ &= \sum_{n=0}^{\infty} \frac{\Delta x_{\beta_1}\Delta x_{\beta_2}\cdots\Delta x_{\beta_{2n+1}}}{2^{2n}\cdot(2n+1)!} f_{,\beta_1\beta_2\cdots\beta_{2n+1}}(\bar{x}) \\ &= \sum_{n=0}^{\infty} \frac{\Delta X_{\beta_1}\Delta X_{\beta_2}\cdots\Delta X_{\beta_{2n+1}}}{2^{2n}\cdot(2n+1)!} F_{\beta_1\beta_2\cdots\beta_{2n+1}}(\bar{x}) \\ &= \Delta X_{D+1}R(\bar{x}) + \sum_{n=1}^{\infty} \frac{\Delta X_{\beta_1}\Delta X_{\beta_2}\cdots\Delta X_{\beta_{2n+1}}}{2^{2n}\cdot(2n+1)!} F_{\beta_1\beta_2\cdots\beta_{2n+1}}(\bar{x}) \end{aligned}\tag{3.4}$$

where

$$F_{\beta_1\beta_2\cdots\beta_{2n+1}}(\bar{x}) = a_{\beta_1\gamma_1}(\bar{x})a_{\beta_2\gamma_2}(\bar{x})\cdots a_{\beta_{2n+1}\gamma_{2n+1}}(\bar{x})f_{,\gamma_1\gamma_2\cdots\gamma_{2n+1}}(\bar{x}).$$

We divide the right-hand side of (3.4) into two parts; one consists of terms with the factor ΔX_{D+1} and the other is the remainder. Since $|\Delta\boldsymbol{X}_j| \sim \sqrt{\hbar\Delta t}$ in the path integral for the

Hamiltonian $\hat{H} = \frac{1}{2}\hat{p}^2 + V(\hat{x})$, the latter is a quantity of the order of magnitude $|\Delta t|^{3/2}$ for small Δt. Thus we can write (3.4) as

$$f(x) - f(x') = \Delta X_{D+1}\{R(\bar{x}) + A(x, x')\} + O(|\Delta t|^{3/2}) \tag{3.5}$$

where $A(x, x')$ takes the form such that

$$A(x, x') = \frac{1}{8}\sum_{j,l=1}^{D} \Delta \boldsymbol{X}_j \Delta \boldsymbol{X}_l F_{D+1jl}(\bar{x}) + \frac{1}{8}\Delta X_{D+1}\sum_{j=1}^{D} \Delta \boldsymbol{X}_j F_{D+1D+1j}(\bar{x})$$
$$+ \frac{1}{24}(\Delta X_{D+1})^2 F_{D+1D+1D+1}(\bar{x}) + \text{higher terms in } \Delta X_{D+1} + O(|\Delta t|^2). \tag{3.6}$$

We may consistently assume that $R(\bar{x}) + A(x, x')$ is non-vanishing for small Δt when regarded as a function of ΔX_{D+1}. Then we are led to

$$\delta(f(x) - f(x')) = \delta(\Delta X_{D+1})\{R(\bar{x}) + A(x, x')\}^{-1} + O(|\Delta t|^{3/2})$$
$$= \delta(\Delta X_{D+1})\left\{R(\bar{x}) + \frac{Q(x, x')}{R(\bar{x})}\right\}^{-1} + O(|\Delta t|^{3/2}) \tag{3.7}$$

where

$$Q(x, x') = \frac{R(\bar{x})}{8}\sum_{j,l=1}^{D} \Delta \boldsymbol{X}_j \Delta \boldsymbol{X}_l F_{D+1jl}(\bar{x})$$
$$= \frac{1}{8}\Delta x_\beta \Delta x_\gamma \Lambda_{\beta\rho}(\bar{x})\Lambda_{\gamma\sigma}(\bar{x}) f_{,\rho\sigma\tau}(\bar{x}) f_{,\tau}(\bar{x}). \tag{3.8}$$

Hence we have

$$\delta(\Delta X_{D+1}) = \delta(f(x) - f(x'))\left\{R(\bar{x}) + \frac{Q(x, x')}{R(\bar{x})}\right\} + O(|\Delta t|^{3/2}) \tag{3.9}$$

and obtain

$$\delta(f(x))\delta(\Delta X_{D+1}) = \delta(f(x))\left\{R(\bar{x}) + \frac{Q(x, x')}{R(\bar{x})}\right\}\delta(f(x')) + O(|\Delta t|^{3/2}) \tag{3.10}$$

which leads us to

$$\delta\big(f(x^{(N)})\big)\prod_{k=1}^{N}\delta\big(\Delta X^{(k)}_{D+1}\big)$$
$$= \prod_{k=1}^{N}\delta(f(x^{(k)})\left\{R(\bar{x}^{(k)}) + \frac{Q(x^{(k)}, x^{(k-1)})}{R(\bar{x})}\right\}\delta(f(x^{(k-1)}) + O(|\Delta t|^{1/2}). \tag{3.11}$$

Inserting this equation into (2.22) we find

$$T_{FI} = \lim_{N\to\infty}\int\prod_{k=0}^{N}\mathrm{d}^{D+1}x^{(k)}\delta(f(x^{(k)}))\prod_{k=1}^{N}\left\{R(\bar{x}^{(k)}) + \frac{Q(x^{(k)}, x^{(k-1)})}{R(\bar{x})}\right\}\frac{\tilde{I}(x^{(k)}, x^{(k-1)})}{(2\pi\hbar)^D}$$
$$\times\psi^*_F(x^{(N)})\psi_I(x^{(0)}). \tag{3.12}$$

Consequently applying (3.3) we arrive at

$$T_{FI} = \lim_{N\to\infty}\int\prod_{k=0}^{N}\mathrm{d}^{D+1}x^{(k)}\delta\big(f(x^{(k)})\big)$$
$$\times\int\prod_{k=1}^{N}\frac{\mathrm{d}^{D+1}p^{(k)}}{(2\pi\hbar)^D}\delta\big(p^{(k)}_\beta f_{,\beta}(\bar{x}^{(k)})\big)\{R^2(\bar{x}^{(k)}) + Q(x^{(k)}, x^{(k-1)})\}$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{k=1}^{N}\{p^{(k)}\cdot\Delta x^{(k)} - H(p^{(k)}, \bar{x}^{(k)})\Delta t\}\right]\psi^*_F(x^{(N)})\psi_I(x^{(0)}) \tag{3.13}$$

which provides us with a rigorous version of path integral representation (3.1). In the above calculation the term $O(|\Delta t|^{1/2})$ in (3.11) has been discarded because of its vanishing contribution in the limit $\Delta t \to 0$. On the other hand, since $Q(x^{(k)}, x^{(k-1)})$ is of the order $O(\hbar \Delta t)$, we expect that it will bring about a finite quantum effect unless $Q \equiv 0$. In this connection it is noted that there is no classical counterpart corresponding to the quantity Q. Furthermore, on account of $\bar{x}^{(k)} = x^{(k)} - \Delta x^{(k)}/2 = x^{(k-1)} + \Delta x^{(k)}/2$, we can write $R(\bar{x}^{(k)})$ as

$$R(\bar{x}^{(k)}) = \tfrac{1}{2}\{R(x^{(k)}) + R(x^{(k-1)})\} + \tfrac{1}{16}\Delta x_\alpha \Delta x_\beta \partial_\alpha \partial_\beta \{R(x^{(k)}) + R(x^{(k-1)})\} + \text{higher terms}. \tag{3.14}$$

Hence for $x^{(k)}$ and $x^{(k-1)}$ on the normalized manifold it takes the form of $\{1 + O(\hbar \Delta t)\}$, which will produce a quantum effect in the second line of (3.13). Accordingly, in the Hamiltonian of (3.13) and in $L_{\text{eff}}^{(k)}$ of (3.15) below we can use $R(\bar{x}^{(k)}) = 1$ by neglecting $O(\hbar \Delta t)$, because in the path integral the quantities H and $L_{\text{eff}}^{(k)}$ always appear in a form multiplied with Δt.

Now let us perform the p-integration in (3.13). The result is obtained by inserting $\tilde{I}(x^{(k)}, x^{(k-1)})$ of (2.19) or (2.21) into the right-hand side of (3.12). Then we have

$$\begin{aligned} T_{FI} = \lim_{N\to\infty} \frac{1}{(2\pi \mathrm{i} \Delta t)^{DN/2}} \int \prod_{k=0}^{N} \mathrm{d}^{D+1} x^{(k)} \delta(f(x^{(k)})) \prod_{k=1}^{N} \left\{ R(\bar{x}^{(k)}) + \frac{Q(x^{(k)}, x^{(k-1)})}{R(\bar{x}^{(k)})} \right\} \\ \times \exp\left[\frac{\mathrm{i}}{\hbar} \Delta t \sum_{k=1}^{N} L_{\text{eff}}^{(k)} \right] \psi_F^*(x^{(N)}) \psi_I(x^{(0)}) \end{aligned} \tag{3.15}$$

where $L_{\text{eff}}^{(k)}$ is given by (2.25) for $D = 1$ and by (2.26) for $D \geqslant 2$.

We will call (3.15) the path integral representation of type II, which is the Lagrangian version of the FS-type path integral. Unlike the case of type I there appears the primary constraint condition $\delta(f(x^{(k)}))$ corresponding to each k.

4. Concluding remarks

We have formulated rigorous expressions for the path integral, which describes the transition amplitude for the system constrained on $f(x) = 0$. The basic equation for this is (2.10), in which the constraint function $f(x)$ is required to satisfy (1.15) and wavefunctions defined on $\mathcal{H}$ are assumed to obey (1.12). Then the two types of path integral representations called types I and II have been formulated in sections 2 and 3, respectively. Though the appearances are quite different they provide us with equivalent descriptions of the same system, since as easily seen each of them have been obtained by rewriting equation (2.10) in an equivalent manner.

In closing this paper we add a few remarks.

The simplest example of our argument is the system on S^D (radius a), where the normalized $f(x)$ is given by

$$f(x) = \frac{(x^2 - r^2)}{2a} \qquad (x^2 \equiv x_\alpha x_\alpha) \tag{4.1}$$

and hence

$$\Lambda_{\beta\gamma}(x) = \delta_{\beta\gamma} - \frac{x_\beta x_\gamma}{x^2} \qquad Q(x) = 0. \tag{4.2}$$

The Hamiltonian $H(p, x)$ in the right-hand side of (3.3) is found to take the form

$$H(p,x)=\begin{cases}\frac{1}{2}\left\{\left(p_1-\alpha\hbar\frac{x_2}{x^2}\right)^2+\left(p_2+\alpha\hbar\frac{x_1}{x^2}\right)^2\right\}+V_{\rm eff}(x) & (D=1)\\ \frac{1}{2}p^2+V_{\rm eff}(x) & (D\geqslant 2)\end{cases} \tag{4.3}$$

with

$$V_{\rm eff}(x)=\hbar^2\frac{D}{8x^2}+V(x). \tag{4.4}$$

As will be expected from the argument in section 2, the first line in the above is just the Aharonov–Bohm Hamiltonian with effective potential $V_{\rm eff}(x)$. In the path integral the term x^2 in the Hamiltonian can be replaced with a^2 by virtue of $R(\bar{x}^{(k)}) = 1$. A detailed study of (3.13) with the Hamiltonian $\frac{1}{2}p^2 + V(x)$ on S^D $(D \geqslant 1)$ has been made by Fukutaka and Kashiwa [9] in their analysis of the FS formula.

As pointed out in I, our formalism can easily be generalized to the case where the manifold $f(x) = 0$ is diffeomorphic to $\mathbb{R}^D$. The Dirac algebra in this case takes the same form as (1.2)–(1.6). Thus the operators $\hat{p}_\beta$ of (1.10) for $D = 1$ and of (1.11) for $D \geqslant 2$ satisfy this algebra. It is noted, however, that the operators $\hat{p}_\beta$ specified with $\alpha \neq 0$ in (1.10) are unitarily equivalent to those with $\alpha = 0$ because of the simply connected structure of the manifold, that is the gauge potentials in (2.27) are eliminated by applying a suitable gauge transformation. Thus, irrespective of the value of D we may write $\hat{p}_\beta$ as

$$\hat{p}_\beta=\tfrac{1}{2}\{\Lambda_{\beta\gamma}(\hat{x}),\hat{\pi}_\gamma\} \tag{4.5}$$

in any irreducible representation. Accordingly the path integral representations, say (2.24) and (3.15), are given with $L^{(k)}_{\rm eff}$ of (2.26) for any D.

Finally we remark that we are unable to represent the trace of $\mathrm{e}^{-\beta\hat{H}}$ in the form of a path integral in the manner stated in the present paper. The reason is as follows.

Since the trace is to be taken on the physical Hilbert space $\underline{\mathcal{H}}$, it should be written as

$$\mathrm{Tr}\,\mathrm{e}^{-\beta\hat{H}}=\sum_{n=1,2,3,\ldots}(\underline{\psi}_n|\,\mathrm{e}^{-\beta\hat{H}}|\underline{\psi}_n) \tag{4.6}$$

with the aid of the complete set of ortho-normalized vectors $|\underline{\psi}_n)$ $(n = 1, 2, 3, \ldots)$ on $\underline{\mathcal{H}}$. Then owing to (1.14) we are led to

$$\mathrm{Tr}\,\mathrm{e}^{-\beta\hat{H}}=\sum_{n=1,2,3,\ldots}\int \mathrm{d}^{D+1}x\,\mathrm{d}^{D+1}x'\,\delta(f(x))\langle x|\mathrm{Tr}\,\mathrm{e}^{-\beta\hat{H}}|x'\rangle\psi_n^*(x)\psi_n(x') \tag{4.7}$$

in which, according to (1.12), square integrable functions $\psi_n(x) \in \mathcal{H}$ are related to $\underline{\psi}_n(\underline{x})$ by

$$\psi_n(x)|_{x=\underline{x}}=\underline{\psi}_n(\underline{x}) \qquad (n=1,2,3,\ldots). \tag{4.8}$$

Therefore, if $\psi_n(x)$ were made to satisfy the ortho-completeness condition

$$\begin{cases}\int \mathrm{d}x^{D+1}\psi_n^*(x)\psi_{n'}(x)=\delta_{nn'}\\ \sum_{n=1,2,\ldots}\psi_n(x)\psi_n^*(x')=\delta^{D+1}(x-x')\end{cases} \tag{4.9}$$

on $\mathcal{H}$, the right-hand side of (4.8) would be written as $\int \mathrm{d}^{D+1}x\delta(f(x))\langle x|\mathrm{Tr}\,\mathrm{e}^{-\beta\hat{H}}|x\rangle$, which could provide us with a path integral representation for the trace. However, the situation is not so simple. For instance, let us consider a particle constrained to move on S^2 (radius a) under the central potential $V_{\rm eff}(r)$, where $r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$. Then $\underline{\psi}_n(\underline{x})$ is represented by the spherical harmonics $Y_l^m(\theta, \varphi)$ with $n = (l, m)$ and $\underline{x} = (\theta, \varphi)$. Hence for reasons of symmetry the auxiliary wavefunction $\psi_n(x)$ should be written as $Y_l^m(\theta, \varphi)F_{lm}(r)$ with $F_{lm}(a) = 1$. It is noted here that in this expression for $\psi_n(x)$ no quantum number exists corresponding to the radial degree of freedom, and accordingly the functions $Y_l^m(\theta,\varphi)F_{lm}(r)$ as a whole cannot span an ortho-complete system on $\mathcal{H}$. This clearly contradicts condition (4.9) which could provide a path integral representation for the trace. Accordingly a study of an alternative approach to attack this problem would be highly desirable.

Appendix A

Let us start by evaluating the quantity $\langle x|\hat{\pi}_\alpha\Lambda_{\alpha\beta}(\hat{x})\hat{\pi}_\beta|x'\rangle$ using

$$\langle x|\hat{\pi}_\alpha|x'\rangle = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_\alpha}\delta^{D+1}(x-x'). \tag{A.1}$$

Then we have

$$\begin{aligned}
\langle x|\hat{\pi}_\alpha\Lambda_{\alpha\beta}(\hat{x})\hat{\pi}_\beta|x'\rangle &= \int \mathrm{d}^{D+1}y\,\mathrm{d}^{D+1}y'\langle x|\hat{\pi}_\alpha|y\rangle\langle y|\Lambda_{\alpha\beta}(\hat{x})|y'\rangle\langle y'|\hat{\pi}_\beta|x'\rangle \\
&= \hbar^2\frac{\partial^2}{\partial x_\alpha\partial x'_\beta}\int \mathrm{d}^{D+1}y\,\mathrm{d}^{D+1}y'\delta^{D+1}(x-y)\delta^{D+1}(y'-x')\delta^{D+1}(y-y') \\
&\quad\times\Lambda_{\alpha\beta}\left(\frac{y+y'}{2}\right) \\
&= \hbar^2\frac{\partial^2}{\partial x_\alpha\partial x'_\beta}\left(\delta^{D+1}(x-x')\Lambda_{\alpha\beta}\left(\frac{x+x'}{2}\right)\right) \\
&= \hbar^2\left(-\partial_\alpha\partial_\beta\delta^{D+1}(x-x')\cdot\Lambda_{\alpha\beta}(\bar{x}) + \frac{1}{4}\delta^{D+1}(x-x')\partial_\alpha\partial_\beta\Lambda_{\alpha\beta}(\bar{x})\right) \\
&\quad+\frac{\hbar^2}{2}\delta^{D+1}(x-x')\left(\frac{\overleftarrow{\partial}}{\partial x_\alpha}\frac{\overrightarrow{\partial}}{\partial x_\beta} - \frac{\overleftarrow{\partial}}{\partial x_\beta}\frac{\overrightarrow{\partial}}{\partial x_\alpha}\right)\Lambda_{\alpha\beta}(\bar{x})
\end{aligned} \tag{A.2}$$

where $\bar{x} = \frac{1}{2}(x+x')$. Since the second term in the right-hand side vanishes due to $\Lambda_{\alpha\beta}(\bar{x}) = \Lambda_{\beta\alpha}(\bar{x})$, we are led to

$$\begin{aligned}
&\frac{1}{2}\langle x^{(k)}|\hat{\pi}_\alpha\Lambda_{\alpha\beta}(\hat{x})\hat{\pi}_\beta|x^{(k-1)}\rangle \\
&\quad= \int\frac{\mathrm{d}^{D+1}p^{(k)}}{(2\pi\hbar)^{D+1}}\exp\left(\frac{\mathrm{i}}{\hbar}p^{(k)}\cdot\Delta x^{(k)}\right)\left\{\frac{1}{2}(p^{(k)\perp})^2 + \frac{\hbar^2}{8}\partial_\alpha\partial_\beta\Lambda_{\alpha\beta}(\bar{x}^{(k)})\right\}
\end{aligned} \tag{A.3}$$

where use has been made of $\Lambda_{\alpha\beta}(\bar{x}^{(k)}) = \Lambda_{\alpha\gamma}(\bar{x}^{(k)})\Lambda_{\gamma\beta}(\bar{x}^{(k)})$. Furthermore, since

$$\langle x^{(k)}|G(\hat{x})|x^{(k-1)}\rangle = \frac{1}{(2\pi\hbar)^{D+1}}\int\mathrm{d}^{D+1}p^{(k)}\exp\left(\frac{\mathrm{i}}{\hbar}p^{(k)}\cdot\Delta x^{(k)}\right)G(\bar{x}) \tag{A.4}$$

for any $G(\hat{x})$, we obtain from (2.2), (2.3) and (2.7)

$$\begin{aligned}
&\langle x^{(k)}|\left(\tfrac{1}{2}\hat{\pi}_\alpha\Lambda_{\alpha\beta}(\hat{x})\hat{\pi}_\beta + K(\hat{x}) + V(\hat{x})\right)|x^{(k-1)}\rangle \\
&\quad= \int\frac{\mathrm{d}^{D+1}p^{(k)}}{(2\pi\hbar)^{D+1}}\exp\left(\frac{\mathrm{i}}{\hbar}p^{(k)}\cdot\Delta x^{(k)}\right) \\
&\qquad\times\left\{\frac{1}{2}(p^{(k)\perp})^2 + \frac{\hbar^2}{8}\partial_\alpha\partial_\beta\Lambda_{\alpha\beta}(\bar{x}^{(k)}) + K(\bar{x}^{(k)}) + V(\bar{x}^{(k)})\right\} \\
&\quad= \int\frac{\mathrm{d}^{D+1}p^{(k)}}{(2\pi\hbar)^{D+1}}\exp\left(\frac{\mathrm{i}}{\hbar}p^{(k)}\cdot\Delta x^{(k)}\right)\left\{\frac{1}{2}(p^{(k)\perp})^2 + V_{\mathrm{eff}}(\bar{x}^{(k)})\right\}.
\end{aligned} \tag{A.5}$$

Thus for $D \geqslant 2$ we are led to (2.4) with the Hamiltonian of (2.6).

Next let us consider the case of $D = 1$, where as seen from (2.1) $\hat{H}$ contains an additional term of the form $\{G_\rho(\hat{x}), \hat{\pi}_\rho\}$. In a similar manner to the above we can easily evaluate $\langle x|\{G_\rho(\hat{x}), \pi_\rho\}|x'\rangle$ to obtain

$$\begin{aligned}
\langle x|\{G_\rho(\hat{x}), \pi_\rho\}|x'\rangle &= \int \mathrm{d}^{D+1}y(\langle x|G_\rho(\hat{x})|y\rangle\langle y|\hat{\pi}_\rho|x'\rangle + \langle x|\pi_\rho|y\rangle\langle y|\hat{G}_\rho(\hat{x})|x'\rangle) \\
&= \mathrm{i}\hbar \int \mathrm{d}^{D+1}y \left(G_\rho\left(\frac{x+y}{2}\right)\delta^{D+1}(x-y)\frac{\partial}{\partial x'_\rho}\delta^{D+1}(y-x') \right. \\
&\quad \left. - G_\rho\left(\frac{y+x'}{2}\right)\delta^{D+1}(y-x')\frac{\partial}{\partial x_\rho}\delta^{D+1}(x-y)\right) \\
&= \mathrm{i}\hbar\left(\frac{\partial}{\partial x'_\rho} - \frac{\partial}{\partial x_\rho}\right) G_\rho(\bar{x})\delta^{D+1}(x-x') \\
&= -2\mathrm{i}\hbar G_\rho(\bar{x})\partial_\rho\delta^{D+1}(x-x') \\
&= \frac{2}{(2\pi\hbar)^{D+1}}\int \mathrm{d}^{D+1}p \exp\left(\frac{\mathrm{i}}{\hbar}p\cdot(x-x')\right)G_\rho(\bar{x})p_\rho.
\end{aligned} \tag{A.6}$$

Hence

$$\begin{aligned}
&\frac{\alpha\hbar}{2}\epsilon_{\beta\tau}\langle x^{(k)}|\left\{\frac{f_{,\beta}(\hat{x})f_{,\tau\sigma}(\hat{x})\Lambda_{\sigma\rho}(\hat{x})}{R^2(\hat{x})}, \hat{\pi}_\rho\right\}|x^{(k-1)}\rangle \\
&\quad = \frac{\alpha\hbar\epsilon_{\beta\tau}}{(2\pi\hbar)^{D+1}}\int \mathrm{d}^{D+1}p^{(k)} \exp\left(\frac{\mathrm{i}}{\hbar}p^{(k)}\cdot\Delta^{(k)}\right)\frac{f_{,\beta}(\bar{x}^{(k)})f_{,\tau\sigma}(\bar{x}^{(k)})p_\sigma^{(k)\perp}}{R^2(\bar{x}^{(k)})}
\end{aligned} \tag{A.7}$$

which gives (2.5) if use is made of (A.4) and (A.6). Thus we have derived (2.5) and (2.6).

The result obtained here just corresponds to the Weyl ordering for the operator product. In this connection it is to be noted that in the above calculation we can replace $\bar{x}$ in the functions $\delta^{D+1}(x-x')\Lambda_{\alpha\beta}(\bar{x})$, $\delta^{D+1}(x-x')G_\rho(\bar{x})$ and $\delta^{D+1}(x-x')G(\bar{x})$ by $\bar{x}^\eta \equiv \eta x + (1-\eta)x'$ with real η. Then we will have different expressions for (A.3) and (A.5), e.g., such that

$$\begin{aligned}
\langle x|\hat{\pi}_\alpha\Lambda_{\alpha\beta}(\hat{x})\hat{\pi}_\beta|x'\rangle &= \frac{1}{(2\pi\hbar)^{D+1}}\int \mathrm{d}^{D+1}p\exp\left(\frac{\mathrm{i}}{\hbar}p\cdot(x-x')\right) \\
&\quad \times (p^{\perp 2} + \hbar^2\eta(1-\eta)\partial_\alpha\partial_\beta\Lambda_{\alpha\beta}(\bar{x}^\eta) + \mathrm{i}\hbar(2\eta-1)p_\alpha\partial_\beta\Lambda_{\alpha\beta}(\bar{x}^\eta)).
\end{aligned} \tag{A.8}$$

Obviously $\eta = 1/2$ provides the simplest expression. For this reason we have employed it in the present paper. The privileged role of $\eta = 1/2$ was emphasized by Fukutaka and Kashiwa [9] in their analysis of the FS formula.

Appendix B

To begin with we note the following identity:

$$A_\beta(\bar{x})\Lambda_{\beta\gamma}(\bar{x})A_\gamma(\bar{x}) = \alpha^2\hbar^2\frac{\Lambda_{\beta\gamma}(\bar{x})f_{,\gamma\sigma}(\bar{x})\Lambda_{\sigma\tau}(\bar{x})f_{,\tau\beta}(\bar{x})}{R^2(\bar{x})} \tag{B.1}$$

where $A_\beta(\bar{x})$ is defined by (2.20) and the right-hand side in the above is just twice the fourth term in the right-hand side of (2.17). A proof of (B.1) is as follows:

$$\begin{aligned}
A_\beta\Lambda_{\beta\gamma}A_\gamma &= \alpha^2\hbar^2\frac{\epsilon_{\sigma\tau}f_{,\sigma}f_{,\tau\beta}}{R^2}\Lambda_{\beta\gamma}\frac{\epsilon_{\kappa\rho}f_{,\kappa}f_{,\rho\gamma}}{R^2} \\
&= \alpha^2\hbar^2(\delta_{\sigma\tau}\delta_{\tau\rho} - \delta_{\sigma\rho}\delta_{\tau\kappa})\Lambda_{\beta\gamma}\frac{f_{,\sigma}f_{,\tau\beta}f_{,\kappa}f_{,\rho\gamma}}{R^4}
\end{aligned}$$

$$= \alpha^2\hbar^2 \left(\Lambda_{\beta\gamma} \frac{f_{,\gamma\tau} f_{,\tau\beta}}{R^2} - \Lambda_{\beta\gamma} \frac{f_{,\sigma} f_{,\tau} f_{,\gamma\sigma} f_{,\tau\beta}}{R^4} \right)$$
$$= \alpha^2\hbar^2 \left(\Lambda_{\beta\gamma} \frac{f_{,\gamma\tau} f_{,\tau\beta}}{R^2} + \Lambda_{\beta\gamma} (\Lambda_{\sigma\tau} - \delta_{\sigma\tau}) \frac{f_{,\gamma\sigma} f_{,\tau\beta}}{R^2} \right)$$
$$= \alpha^2\hbar^2 \frac{\Lambda_{\beta\gamma} f_{,\gamma\sigma} \Lambda_{\sigma\tau} f_{,\tau\beta}}{R^2} \tag{B.2}$$

where we have omitted writing the argument $\bar{x}$ for simplicity.

Accordingly, we can now write $H(\boldsymbol{P}, \bar{x})$ in (2.17) as

$$H(\boldsymbol{P}, \bar{x}) = \tfrac{1}{2}\{\boldsymbol{P}^2 - 2A_\beta(\bar{x}) a_{1\beta}(\bar{x}) \boldsymbol{P} + A_\beta(\bar{x}) \Lambda_{\beta\gamma}(\bar{x}) A_\gamma(\bar{x})\} + V_{\text{eff}}(\bar{x}) \tag{B.3}$$

which leads us to

$$\boldsymbol{P}\Delta\boldsymbol{X} - H(\boldsymbol{P}, \bar{x})\Delta t = -\frac{1}{2}\left(\boldsymbol{P} - A_\beta(\bar{x}) a_{1\beta}(\bar{x}) - \frac{\Delta\boldsymbol{X}}{\Delta t} \right)^2 \Delta t$$
$$+ \left\{ \frac{1}{2} \left(\frac{\Delta\boldsymbol{X}}{\Delta t} \right)^2 + A_\beta(\bar{x}) a_{1\beta}(\bar{x}) \frac{\Delta\boldsymbol{X}}{\Delta t} - V_{\text{eff}}(\bar{x}) \right\} \Delta t \tag{B.4}$$

where we have used the relation $a_{1\beta}(\bar{x}) a_{1\gamma}(\bar{x}) = \Lambda_{\beta\gamma}(\bar{x})$, which comes from

$$\| a_{\beta\gamma}(\bar{x}) \| = \begin{pmatrix} \dfrac{f_{,2}(\bar{x})}{R(\bar{x})} & -\dfrac{f_{,1}(\bar{x})}{R(\bar{x})} \\ \dfrac{f_{,1}(\bar{x})}{R(\bar{x})} & \dfrac{f_{,2}(\bar{x})}{R(\bar{x})} \end{pmatrix}. \tag{B.5}$$

Since $\Delta\boldsymbol{X} = \Delta X_1$, we find

$$(\Delta\boldsymbol{X})^2 = \Delta X_\beta \Delta X_\beta |_{\Delta X_2 = 0} = \Delta x_\beta \Delta x_\beta |_{\Delta X_2 = 0} \tag{B.6}$$

and

$$a_{1\beta}(\bar{x}) \Delta\boldsymbol{X} = a_{\sigma\beta}(\bar{x}) \Delta X_\sigma |_{\Delta X_2 = 0} = \Delta x_\beta |_{\Delta X_2 = 0}. \tag{B.7}$$

Using (B.4) together with (B.5) and (B.6) in (2.15) we then obtain for $D = 1$

$$\tilde{I}(x, x') = \int \mathrm{d}\boldsymbol{P} \exp\left[-\frac{\mathrm{i}}{2\hbar} \boldsymbol{P}^2 \Delta t \right] \exp\left[\frac{\mathrm{i}}{\hbar} \left\{ \frac{1}{2} \left(\frac{\Delta x_\beta}{\Delta t} \right)^2 + A_\beta(\bar{x}) \frac{\Delta x_\beta}{\Delta t} - V_{\text{eff}}(\bar{x}) \right\} \Delta t \right] \Bigg|_{\Delta X_2 = 0}$$
$$= \left(\frac{2\pi\hbar}{\mathrm{i}\Delta t} \right)^{1/2} \exp\left[\frac{\mathrm{i}}{\hbar} \left\{ \frac{1}{2} \left(\frac{\Delta x_\beta}{\Delta t} \right)^2 + A_\beta(\bar{x}) \frac{\Delta x_\beta}{\Delta t} - V_{\text{eff}}(\bar{x}) \right\} \Delta t \right] \Bigg|_{\Delta X_2 = 0}. \tag{B.8}$$

Since as seen from (2.14) the term $\tilde{I}(x, x')$ always appears in the form multiplied by $\delta(\Delta X_{D+1})$ we can effectively remove the condition $\Delta X_2 = 0$ from (B.8), thereby obtaining (2.19).

In a similar manner we can derive (2.21) for $D \geqslant 2$.

References

[1] Dirac P A M 1950 *Can. J. Math.* **2** 129
Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science, Yeshiva University)

[2] Ohnuki Y 2003 *J. Phys. A: Math. Gen.* **36** 6509

[3] See, for example, Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)
Also Kashiwa T, Ohnuki Y and Suzuki M 1997 *Path Integral Methods* (Oxford: Clarendon)

[4] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485

[5] Faddeev L D 1970 *Theor. Math. Phys.* **1** 1

[6] Senjanovic P 1976 *Ann. Phys., NY* **100** 227
[7] Schulman L S 1971 *J. Math. Phys.* **12** 305
[8] Ohnuki Y 1987 *Proc. 2nd Int. Symp. Foundation of Quantum Mechanics* ed M Namiki *et al* (Tokyo: Physical Society of Japan) p 117
[9] Fukutaka H and Kashiwa T 1987 *Ann. Phys., NY* **175** 30
Fukutaka H and Kashiwa T 1988 *Prog. Theor. Phys.* **80** 301
[10] Kashiwa T 1996 *Prog. Theor. Phys.* **95** 431